UNIVERSITY OF BOLOGNA

Example of application of pairwise comparison to identify the optimal dam site

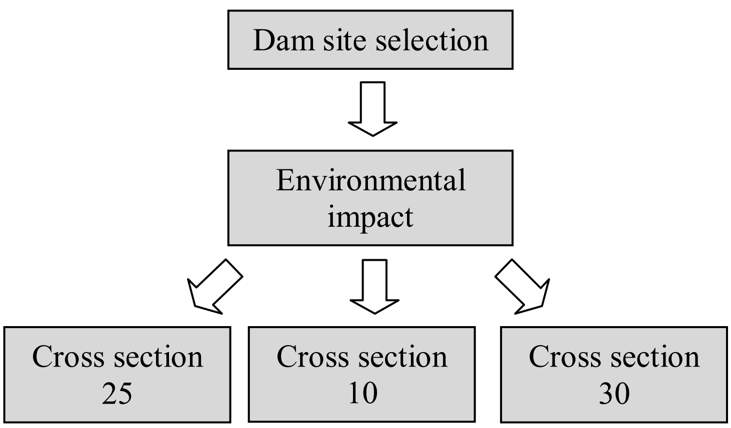
This case study describes the use of the Analytical Hierarchy Process (AHP) in choosing the optimal site for building a dam. There are three possible alternatives along the same river. By using the AHP, the decision maker is able to choose the best site in a rational and transparent way that can be examined and understood by all concerned.

The table below shows the information available on the three sites.

		Dam location		
		River cross section 10	**River cross section 25**	**River cross section 30**
Criteria	Estimated cost (billion euros)	45	30	55
	Reservoir storage (10^6 m^3)	50	25	30
	Environmental impact	Very heavy impact on wildlife and river ecology	Valuable forests submerged with ecosystems impacted	The impact is limited to lowland areas of limited environmental value
	Landscape impact	The landscape is not impacted	There is a significant impact on landscape due to forests disappearing	The impact on the landscape is limited

Let's examine the case where one criteria only is used to select the optimal site. Let us assume that the environmental impact is the selected criteria. Such criteria is not immediately related to a numerical benefit function, being the association of environmental value to an economical indicator extremely difficult. In such a case, where the alternatives are limited, the AHP process, through pairwise comparison, is an interesting opportunity.

The AHP hierarchy for this decision is shown below.



Pairwise comparisons

The priorities will be derived from a series of measurements: pairwise comparisons involving all the nodes. The nodes at each level will be compared, two by two, with respect to their contribution to the nodes above them. The results of these comparisons will be entered into a matrix which is processed mathematically to derive the priorities for all the nodes on the level. The comparisons can be made in any sequence, but in this example we will begin by comparing the Alternatives with respect to their strengths in meeting each the Criteria. Since there are three Alternatives and we need to compare each one to each of the others, we will make three pairwise comparisons with respect to each Criterion: Cross Section 25 versus Cross Section 10, 10 against 30 and 30 against 25. For each comparison, the decision maker will first judge which member of the pair is weaker with respect to the Criterion under consideration. Then a relative weight is computed with respect to the other sites.

They will use the AHP Fundamental Scale in assigning the weights:

The Fundamental Scale for Pairwise Comparisons		
Intensity of Importance	**Definition**	**Explanation**
1	Equal importance	Two elements contribute equally to the objective
3	Moderate importance	Experience and judgment moderately favor one element over another
5	Strong importance	Experience and judgment strongly favor one element over another
7	Very strong importance	One element is favored very strongly over another; its dominance is demonstrated in practice
9	Extreme importance	The evidence favoring one element over another is of the highest possible order of affirmation
Intensities of 2, 4, 6, and 8 can be used to express intermediate values. Intensities of 1.1, 1.2, 1.3, etc. can be used for elements that are very close in importance.		

Alternative sites compared with respect to environmental impact:

Site	Mark	Site	Mark	Comment
Cross section 10	1	Cross section 25	3	The lost of forests is considered less impacting than the loss of wildlife and river ecological value
Cross section 10	1	Cross section 30	7	The lowland impact is negligible
Cross Section 25	1	Cross section 30	5	The lowland impact is negligible

The next step is to transfer the weights to a matrix, using a method unique to the AHP. For each pairwise comparison, the number representing the greater weight is transferred to the related box; and the reciprocal of that number is put into the symmetric box:

	Cross section 10	Cross section 25	Cross section 30
Cross section 10	1	1/3	1/7
Cross section 25	3	1	1/5
Cross section 30	7	5	1

By processing this matrix mathematically, the AHP derives priorities for the candidates with respect to Experience. The priorities are measurements of their relative strengths, derived from the judgments of the decision makers as entered into the matrix. Mathematically speaking, they are the values in the matrix's principal right eigenvector. These values can be calculated in many ways, including by hand, or with a spreadsheet program, or by using R. They are shown below to the right of the matrix, along with an Inconsistency Factor computed by the specialized AHP software that was used to process. The computation of the inconsistency factor is explained below.

From R:

```
> pr1=c(1,1/3,1/7,3,1,1/5,7,5,1)
> pr1=array(pr1,dim=c(3,3))
> pr1=t(pr1)
> pr1
     [,1]      [,2]      [,3]
[1,]    1 0.3333333 0.1428571
[2,]    3 1.0000000 0.2000000
[3,]    7 5.0000000 1.0000000
> pr2=eigen(pr1,symmetric="F")
> pesi=Mod(pr2$vectors)[,1]/sum(Mod(pr2$vectors)[,1])
> sum(pesi)
[1] 1
> pesi
[1] 0.08096123 0.18839410 0.73064467
```

The results show that cross section 30 is assigned a weight of **0.73**, versus lower weights of **0.19** and **0.08** for cross section 10 and 25, respectively.

Checking for consistency of the evaluation

In an ideal case where the comparison matrix A is fully consistent, the rank(A) = 1 and $\lambda = n$ (n = number of criteria). In this case, the following equation is valid:

$A \times x = n \times x$ (where x is the eigenvector of A) and the vector x represents the weights we are looking for.

In the non-consistent case (which is more common) the comparison matrix A may be considered as a perturbation of the previous consistent case. When the entries A_{ij} changes only slightly, then the eigenvalues change in a similar fashion. Moreover, the maximum eigenvalue (λmax) is closely grater to n while the remaining (possible) eigenvalues are close to zero. Thus is order to find weights we are looking for the eigenvector which corresponds to the maximum eigenvalue (λmax).

In order to obtain weights from calculated eigenvector the values have to be normalised by the formula below, as the weights have to sum up to 1. The nornalisation, which has been done in the above R code, reads as:

$$w_j = \frac{\tilde{w}_j}{\sum_{i=1}^{n} \tilde{w}_i}$$

The consistency index (CI) is calculated as follows:

$$CI = \frac{\lambda_{\max} - n}{n-1}$$

Then, the consistence ratio (CR) is calculated as the ratio of consistency index and random consistency index (RI).

$$CR(A) = \frac{CI(A)}{RI(n)}$$

The RI is the random index representing the consistency of a randomly generated pairwise comparison matrix It is derived as average random consistency index (Table 1) calculated from a sample of 500 of randomly generated matrices. It only depends on the matrix's size and takes the values shown below:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
RCI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45

If CR(A) ≤ 0.1, the pairwise comparison matrix is considered to be consistent enough.

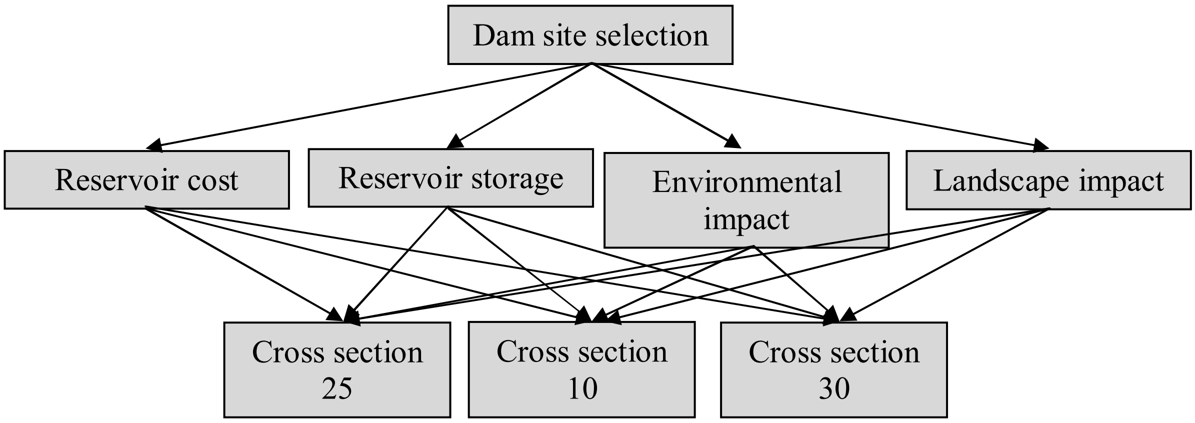
For the above case:

```
> Mod(pr2$values)
[1] 3.0648876 0.4459519 0.4459519
> C1=(max(Mod(pr2$values))-3)/2
> C1
[1] 0.03244379
> CR=C1/0.58
> CR
[1] 0.05593757
```

CR is lower than 0.1 and therefore the evaluation is consistent.

Let's now examine the case where the three criteria are used to select the optimal site. In such a situation, we not only have to find the weight given to each site for each criteria, but we also have to assign a weight to the criteria. We can use pairwise comparison for this purpose as well.

The AHP hierarchy for this decision is shown below.



Pairwise comparisons

		Dam location		
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Alternative sites compared with respect to reservoir cost:

Site	Mark	Site	Mark	Comment
Cross section 10	1	Cross section 25	5	Weights are assigned by looking at quantitative assessment
Cross section 10	3	Cross section 30	1	
Cross Section 25	9	Cross section 30	1	

	Cross section 10	Cross section 25	Cross section 30
Cross section 10	1	1/5	3
Cross section 25	5	1	9
Cross section 30	1/3	1/9	1

> pesi
[1] 0.17817773 0.75140459 0.07041769

The results show that cross section 10 is assigned a weight of **0.18** versus weights of **0.75** and **0.07** for cross section 25 and 30, respectively.

> CR
[1] 0.02505497

CR is lower than 0.1 and therefore the evaluation is consistent.

Alternative sites compared with respect to reservoir storage:

Site	Mark	Site	Mark	Comment
Cross section 10	9	Cross section 25	1	Weights are assigned by looking at quantitative assessment
Cross section 10	7	Cross section 30	1	
Cross Section 25	1	Cross section 30	3	

	Cross section 10	Cross section 25	Cross section 30
Cross section 10	1	9	7
Cross section 25	1/9	1	1/3
Cross section 30	1/7	3	1

> pesi
[1] 0.78539119 0.06579374 0.14881507

The results show that cross section 10 is assigned a weight of **0.79** versus weights of **0.06** and **0.15** for cross section 25 and 30, respectively.

> CR
[1] 0.069224

CR is lower than 0.1 and therefore the evaluation is consistent.

Alternative sites compared with respect to landscape impact:

Site	Mark	Site	Mark	Comment
Cross section 10	7	Cross section 25	1	Weights are assigned by looking at the above descriptiions
Cross section	5	Cross section	1	

10		30		
Cross Section 25	1	Cross section 30	5	

	Cross section 10	Cross section 25	Cross section 30
Cross section 10	1	7	5
Cross section 25	1/7	1	1/5
Cross section 30	1/5	5	1

```
> pesi
[1] 0.71470956 0.06679607 0.21849437
```

The results show that cross section 10 is assigned a weight of **0.71** versus weights of **0.07** and **0.22** for cross section 25 and 30, respectively.

```
> CR
[1] 0.1575576
```

CR is higher than 0.1. Therefore the evaluation is not very consistent. We may consider that we are close to consistency and keep the above weights, or we may decide to remake the assessment. For the purpose of this exercise, we decide to keep the computed values.

Criteria compared each other:

	Estimated cost	Reservoir storage	Environmental impact	Landscape impact
Estimated cost	1	3	3	5
Reservoir storage	1/3	1	1	3
Environmental impact	1/3	1	1	3
Landscape impact	1/5	1/3	1/3	1

```
> pesi
[1] 0.52224472 0.19983200 0.19983200 0.07809127
```

The results show that Estimated cost is assigned a weight of **0.52** versus weights of **0.20, 0.20** and **0.08** for reservoir storage, environmental impact and landscape impact, respectively.

```
> CR
[1] 0.01610869
```

CR is lower than 0.1 and therefore the evaluation is consistent.

Computation of the overall weights

Cross section 10:
W = 0.19 * 0.20 + 0.18 * 0.52 + 0.79 * 0.20 + 0.71 * 0.08 = 0.35

Cross section 25:
W = 0.08 * 0.20 + 0.75 * 0.52 + 0.06 * 0.20 + 0.07 * 0.08 = 0.42

Cross section 30:
W = 0.73 * 0.20 + 0.07 * 0.52 + 0.15 * 0.20 + 0.22 * 0.08 = 0.23

In conclusion, the best alternative is cross section 25.